Algebraic Geometry: MIDTERM SOLUTIONS

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ABSTRACT. Algebraic Geometry: MIDTERM 6^{th} March 2013. We give terse solutions to this Midterm Exam.

1. Problem 1:

PROBLEM 1 (Geometry 1). *When is a ring called a reduced ring? Define integral domain. Let R be a ring. Show that if R is an integral domain then it is reduced. Is the converse true?*

SOLUTION 1. *We say is ring R is reduced if it has no non-zero nilpotents. i.e $a \in R$ with $a^n = 0$ for some positive integer $n \geq 1$ implies $a = 0$.*

In an integral domain there are no zero divisors hence there are no nilpotents either because if $a^n = 0$ and $a \neq 0$ for least postive n then $n > 1$ and hence $a^{n-1}.a = 0$ with $a^{n-1} \neq 0$ which contradicts the property of being an integral domain.

Now for the converse consider the ring $R = \mathbb{Z}/6\mathbb{Z}$. This is clearly reduced but is not an integral domain.

2. Problem 2:

PROBLEM 2 (Geometry 2). *Let I denote the defining ideal of an algebraic set X in the affine $n-$space $\mathbb{A}^n_k$ where k is an algebraically closed field. Then the coordinate ring of X is $R = k[X_1, X_2, \ldots, X_n]/I$. Show that the ring is reduced.*

SOLUTION 2. *The defining ideal I has the property that it is a radical ideal. i.e. if f^n vanishes on X then f itself vanishes on X. Hence R is reduced. Furthermore if S is any set the any subring of $R_S = \{f : S \longrightarrow k\}$ of the ring of functions on S is reduced where k is a reduced ring.*

3. Problem 3:

PROBLEM 3 (Geometry 3). *Let B be a ring and A a subring of B. Let P be a prime ideal in A and $S = A \backslash P$. Let K be the field of fractions of $\frac{A}{P}$.*

1. *Show that $\frac{S^{-1}A}{PS^{-1}A}$ is isomorphic to K.*
2. *Show that $B \otimes_A K$is isomorphic to $\frac{S^{-1}B}{PS^{-1}B}$.*

SOLUTION 3. *First we observe the following.*

OBSERVATION 1. *(1) If*

$$0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$$

is an exact sequence of $A-$modules. Let $S \subset A$ be a multiplicatively closed set. Then the sequence

$$0 \longrightarrow S^{-1}K \longrightarrow S^{-1}M \longrightarrow S^{-1}N \longrightarrow 0$$

is also an exact sequence. Exactness has to do with sets here which are kernels (0-element is unique) and images of maps and not with the module structure. (Depending on the type of maps which in here is both $A-$ and $S^{-1}A-$ module maps we have exactness as a sequence of modules over both rings).

(2) We have an isomorphism $S^{-1}A \otimes_A M \xrightarrow{\cong} S^{-1}M$ as both A and $S^{-1}A-$modules. The elements of $S^{-1}A \otimes_A M$ can be expressed as single tensors instead of sum of certain tensors. Using universal property for the existence of the map one can also show injectivity and surjectivity of the map.

(3) The ring $S^{-1}A$ as an $A-$module is flat. (The ring $S^{-1}A$ is trivially flat over itself.)

(4) Let $f : A \longrightarrow B$ be a homomorphism of rings. Let $S \subset A$ be multiplicatively closed. Then $f(S) = T$ is a multiplicative closed subset of B. Both the algebraic objects $S^{-1}B$ and $T^{-1}B$ are not only rings but are also isomorphic by an obvious map.

$$S^{-1}B \cong T^{-1}B$$

Consider an exact sequence of $A-$modules

$$0 \longrightarrow P \longrightarrow A \longrightarrow \frac{A}{P} \longrightarrow 0$$

Let $S = A \backslash P$. We observe the $(A, S^{-1}A)$ module $S^{-1}P$ sits inside $S^{-1}A$ as an ideal in $S^{-1}A$ and so we have

$$\frac{S^{-1}A}{S^{-1}P} \cong S^{-1}\left(\frac{A}{P}\right) \cong T^{-1}\left(\frac{A}{P}\right) \cong K$$

as rings and not as just $(A, S^{-1}A)-$ modules and the map $S^{-1}A \longrightarrow S^{-1}\frac{A}{P}$ is also a ring map. Here $T = (\frac{A}{P})\backslash 0$ a multiplicatively closed subset of $\frac{A}{P}$.
We have an exact sequence

$$0 \longrightarrow S^{-1}P \longrightarrow S^{-1}A \longrightarrow S^{-1}\left(\frac{A}{P}\right) = K \longrightarrow 0$$

where $S = A \backslash P$. Now we tensor the sequence over A with the ring B as an $A-$module to get a complex

$$S^{-1}P \otimes_A B \longrightarrow S^{-1}A \otimes_A B = S^{-1}B \longrightarrow S^{-1}\left(\frac{A}{P}\right) \otimes_A B = K \otimes_A B \longrightarrow 0.$$

The image of $S^{-1}P \otimes_A B$ surjects onto an extended ideal of $S^{-1}P$ (which is an ideal in $S^{-1}A$) in the bigger ring $S^{-1}B = S^{-1}A \otimes_A B \supset S^{-1}A$ as $A \subset B$. The map $S^{-1}B = S^{-1}A \otimes_A B \longrightarrow S^{-1}\left(\frac{A}{P}\right) \otimes_A B$ is a homomorphism of rings with kernel *exactly equal to the extended ideal of $S^{-1}P$ in $S^{-1}B$. Now $S^{-1}P$ is the extended ideal of $P \subset A$ in $S^{-1}A$. Hence we have $B \otimes_A K \cong \frac{S^{-1}B}{PS^{-1}B}$. This proves the problem.*

4. Problem 4:

PROBLEM 4 (Geometry 4). *Let R be a ring. Let $f : A \longrightarrow B, g : B \longrightarrow C$ be $R-$module homomorphisms. When is the sequence $A \xrightarrow{f} B \xrightarrow{g} C$ called a complex? When is it called exact? Let F be a (finitely generated) free $R-$module and*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be an exact sequence of $R-$modules. Show that:

(1) $\mathrm{Hom}_R(M \oplus N, A) \cong \mathrm{Hom}_R(M, A) \oplus \mathrm{Hom}_R(N, A)$ *for any $R-$modules M and N.*
(2) $0 \longrightarrow \mathrm{Hom}_R(F, A) \longrightarrow \mathrm{Hom}_R(F, B) \longrightarrow \mathrm{Hom}_R(F, C) \longrightarrow 0$ *is exact.*

SOLUTION 4. *The sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is called a complex if $g \circ f = 0$. This sequence of modules is called exact if $ker(g) = Im(f) \subset B$.*

(1) *The isomorphism is given by*

$$(f \oplus g : M \oplus N \longrightarrow A) \longrightarrow (f : M \longrightarrow A, g : N \longrightarrow A).$$

(2) *This sequence $0 \longrightarrow \mathrm{Hom}_R(F, A) \longrightarrow \mathrm{Hom}_R(F, B) \longrightarrow \mathrm{Hom}_R(F, C) \longrightarrow 0$ is exact because it is the same exact sequence repeated $rank(F)$ times and the exact sequence is also same as*

$$0 \longrightarrow A^{\oplus Rank(F)} \longrightarrow B^{\oplus Rank(F)} \longrightarrow C^{\oplus Rank(F)} \longrightarrow 0.$$

5. Problem 5:

PROBLEM 5 (Geometry 5). *Let A be an integral domain. When is A said to be a normal domain? Let $A \subset B$ be domains and $\alpha \in B$ be integral over A. Let K be the field of fractions of A and assume that $K(\alpha)/K$ is a separable extension. Show that the minimal polynomial of α over K have coefficients in A.*

SOLUTION 5. *If the domain A is integrally closed in its field of fractions then it is a normal domain. Unique Factorization Domains are examples of domains which are normal domains. Also an example which is not an unique factorization domain but normal is $\frac{\mathbb{C}[X,Y]}{(Y^2 - X^3 + X)}$. Now we obseve the following.*

(1) *The minimum polynomial f of α over K of degree $n = [K(\alpha) : K]$ has distinct roots $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_n$ also $f(X) = \prod_{i=1}^{n}(X - \alpha_i)$.*
(2) *Every polynomial satisfied by α over K also has roots $\alpha_i : 2 \leq i \leq n$. So all the roots $\alpha_i : 2 \leq i \leq n$ are also integral.*
(3) *Hence the coefficients of f which are elementary symmetric polynomials in $\alpha_i : 1 \leq i \leq n$ are elements of K which are also integral over the ring A.*
(4) *Since A is integrally closed in its field of fractions the coefficients lie in A itself i.e. $f \in A[X]$.*

This proves the problem.

6. Problem 6:

PROBLEM 6 (Geometry 6). *Let k be an algebraically closed field. Let $\mathbb{X} \subset \mathbb{A}^n_k, \mathbb{Y} \subset \mathbb{A}^m_k$ be affine algebraic varieties. Let $F : X \longrightarrow Y$ be a map. When is F said to be a morphism of algebraic varieties. Show that there exists polynomials $f_i : 1 \leq i \leq m \in k[X_1, \ldots, X_n]$ such that for all $a = (a_1, \ldots, a_n) \in \mathbb{X}$ the $m-$tuple $F(a) = (f_i(a) : 1 \leq i \leq m)$.*

SOLUTION 6. *We begin with the notion of regularity. Let $a \in \mathbb{X}$. We say a function $f : \mathbb{X} \longrightarrow k$ is regular at a if there exists a zarsiki open set $U \subset \mathbb{A}^n_k$ and a pair of polynomials $G, H \in k[X_1, \ldots, X_n]$ such that $G \neq 0$ on $U \cap \mathbb{X}, a \in U$ and $f = \frac{H}{G}$ on $U \cap \mathbb{X}$. We say that f is regular function if f is regular at all points of a.*
We say that a function $F : \mathbb{X} \longrightarrow \mathbb{Y}$ is a morphism of varieties if for any regular function $g : \mathbb{Y} \longrightarrow k$ the function $g \circ F : \mathbb{X} \longrightarrow k$ is also a regular function on the variety $\mathbb{X}$.
Now we prove that the morphisms between affine varieties are given by polynomial maps. It is standard exercise though. We sketch the proof. First we classify regular functions $\{f : \mathbb{X} \longrightarrow k\}$.

(1) $\boxed{\textit{Local Expressions}:}$ *Let*
$$(a \in U \cap \mathbb{X}, G, H)$$
denote a local expression for f. G denotes a local denominator for f, H denotes a local numerator of f, $U \cap \mathbb{X}$ is the corresponding local open set for the point a where $f = \frac{H}{G}$ makes sense.

(2) $\boxed{\textit{Nullstellensatz and Partition of Identity}:}$ *The ideal of local denominators for the points in $\mathbb{X}$ generates a unit ideal in the ring $\frac{k[X_1, X_2, \ldots, X_k]}{I(\mathbb{X})}$. This is obvious by "Nullstellensatz" for algebraically closed fields k and not for non-algebraically closed fields (it is not true). The ideal of local denominators have no common zero in $\mathbb{X}$. Over the field of real numbers the function $x^2 + y^2 + 1$ has no zero in $\mathbb{X} = \mathbb{R}^2$ but it does not generate unit ideal $\mathbb{R}[x, y]$. Consider the function*
$$f : \mathbb{R}^2 \longrightarrow \mathbb{R} \text{ given by } f(x, y) = \frac{1}{x^2 + y^2 + 1}.$$
So let
$$1_{\mathbb{X}} = \sum_{i=1}^{n} G_i \mid_{\mathbb{X}} R_i \mid_{\mathbb{X}}$$
for some finitely many local denominators G_i from the local expressions
$$(a_i \in \big(U_i = \{G_i \neq 0\}\big) \cap \mathbb{X}, G_i, H_i)$$
without loss of generality and $R_i \in k[X_1, X_2, \ldots, X_k]$.

(3) $\boxed{\textit{Patching}:}$ *Now $G_i f = H_i$ on $U_i \cap \mathbb{X}$ for $i = 1, \ldots, n$. But then using the partition of identity expression above we conclude two facts.*
   (a) *The patches $U_i \cap \mathbb{X}$ cover $\mathbb{X}$ as all the G_i vanish on $\mathbb{X} \backslash \left(\bigcup\limits_{i=1}^{n} (U_i \cap \mathbb{X}) \right)$.*
   (b) *Also $(\sum\limits_{i=1}^{n} G_i \mid_{\mathbb{X}} R_i \mid_{\mathbb{X}}) f = \sum\limits_{i=1}^{n} H_i \mid_{\mathbb{X}} R_i \mid_{\mathbb{X}}$ at each point of $\mathbb{X}$.*

Hence $f = \sum_{i=1}^{n} H_i \mid_{\mathbb{X}} R_i \mid_{\mathbb{X}}$ *given by a polynomial.*

(4) $\boxed{\textit{Conclusion}:}$ *We conclude two points.*

(a) *Regular functions on* $\mathbb{X}$ *are just polynomial functions on* $\mathbb{X}$.

(b) *Every regular function on* $\mathbb{X}$ *has global extension to* $\mathbb{A}_k^n$ *as a polynomial.*

Consider a morphism of varieties $F : \mathbb{X} \subset \mathbb{A}_k^n \longrightarrow \mathbb{Y} \subset \mathbb{A}_k^m$. *Let* $\pi_i : \mathbb{A}_k^m \longrightarrow k$ *be the* $i^{th}-$*coordinate projection for* $1 \leq i \leq m$. *So we get that* $F \circ \pi_i : \mathbb{X} \longrightarrow k$ *is a regular function. Therefore there exists polynomial functions* $f_i : \mathbb{X} \longrightarrow k$ *such that* $F = (f_i : 1 \leq i \leq m)$. *Conversely if* $F = (f_i = H_i \mid_{\mathbb{X}}: 1 \leq i \leq m)$ *and* $g = G \mid_{\mathbb{Y}}: \mathbb{Y} \longrightarrow k$ *is a regular function then* $g \circ F = G(H_1, H_2, \ldots, H_m) \mid_{\mathbb{X}}$ *is also a polynomial function. Here* $G \in k[X_1, X_2, \ldots, X_m], H_i \in k[X_1, X_2, \ldots, X_n]$ *for* $1 \leq i \leq m$. *This proves the problem.*

7. Problem 7:

Problem 7 (Geometry 7). *Prove or Disprove: Let* $\phi : A \longrightarrow B$ *be a homomorphism of rings. Let* m *be a maximal ideal of* B *then* $\phi^{-1}(m)$ *is a maximal ideal of* A.

Solution 7.

(1) *Let* $A = \mathbb{Z}$, $B = \mathbb{Q}$ *and* ϕ *is the inclusion. Take* $m = (0)$. *Here* $\phi^{-1}(0) = (0)$ *is the unique non-maximal prime ideal in* $\mathbb{Z}$.

(2) *Let* $A = \mathbb{Z}$, $B = \mathbb{Z}/p\mathbb{Z}$ *where* p *is a prime and* ϕ *is the reduction* $\mod p$. *Take* $m = (0)$. *Here* $\phi^{-1}(0) = p\mathbb{Z}$ *is a maximal ideal of* $\mathbb{Z}$.

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